

# Some Aspects of Minimal Length Quantum Mechanics

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## Abstract

String theory, quantum geometry, loop quantum gravity and black hole physics all indicate the existence of a minimal observable length on the order of Planck length. This feature leads to a modification of Heisenberg uncertainty principle. Such a modified Heisenberg uncertainty principle is referred as gravitational uncertainty principle(GUP) in literatures. This proposal has some novel implications on various domains of theoretical physics. Here, we study some consequences of GUP in the spirit of Quantum mechanics. We consider two problem: a particle in an one-dimensional box and momentum space wave function for a "free particle". In each case we will solve corresponding perturbational equations and compare the results with ordinary solutions.

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**Key Words:** Quantum Gravity, Generalized Uncertainty Principle, Generalized Schrödinger equation, Momentum Space Wave Function

# 1 Introduction

The problem of reconciling Quantum Mechanics with General Relativity is one of the task of modern theoretical physics which, until now, has not yet found a consistent and satisfactory solution. Although a full description of quantum gravity is not yet available, there are some general features that seem to go hand in hand with all promising candidates for such a theory where one of them is the existence of a minimal length scale. In other words, one of the most interesting consequences of unification of gravity and quantum mechanics is that in resulting quantum gravity there exists a minimal observable distance on the order of the Planck length,  $l_P = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-33}cm$ , where  $G$  is the Newton constant. The existence of such a fundamental length is a dynamical phenomenon due to the fact that, at Planck scale, there are fluctuations of the background metric, i.e. a limit of the order of Planck length appears when quantum fluctuations of the gravitational field are taken into account. In the language of string theory one can say that a string cannot probe distances smaller than its length. The existence of such minimal observable length which is motivated from string theory[1-7], loop quantum gravity[8], quantum geometry[9] and black holes physics[10], leads to a generalization of Heisenberg uncertainty principle to incorporate gravitational induced uncertainty from very beginning. This feature constitutes a part of the motivation to study the effects of this modified algebra on various observables. Consequences of such a gravitational uncertainty principle (GUP), have been studied extensively[11-20]. The generalization of Schrödinger equation has been considered by Hossenfelder *et al*[21]. Also generalized Schrödinger equation and Hydrogen spectrum in the framework of GUP have been considered by Brau[22] and Akhoury and Yao[23]. Momentum space representation of quantum states has been considered by Kempf *et al*[24]. Here we proceed some more step in this direction. We consider some well-known topics in usual quantum mechanics and re-examine them in the framework of GUP. The problems of a particle in an one-dimensional box and momentum space wave function(in a different view relative to Kempf *et al* point of view) are considered and their generalization in GUP are discussed.

## 2 Minimal Length and GUP

The emergence of a minimal observable distance yields to the generalized uncertainty principle

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha l_P^2 \frac{\Delta p}{\hbar}, \quad (1)$$

where  $\alpha$  is GUP parameter which can be determined from fundamental theory(maybe string theory!)[6]. At energy much below the Planck mass,  $m_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{19} GeV/c^2$ , the extra term in right hand side of equation (1) is irrelevant and the Heisenberg relation is recovered, while, as we approach the Planck energy, this term becomes relevant and, as has been said, it is related to the minimal observable length. A simple calculation gives  $(\Delta x)_{min} = 2l_P\sqrt{\alpha}$  for this minimum length scale. Now the generalized commutation relation becomes,

$$[x, p] = i\hbar(1 + \beta p^2), \quad (2)$$

where  $\beta$  (related to  $\alpha$ ) is a constant restricted to the condition:  $0 \leq \beta \leq 1$ . The case  $\beta \rightarrow 0$  gives the usual quantum mechanics regime while  $\beta \rightarrow 1$  is extreme quantum gravity limit. It is important to note that GUP itself has a perturbational nature and one can consider its more generalized form[24]. Since we are dealing with dynamics, the present form of GUP as equation (1) is more suitable as our primary input.

## 3 A particle in an One-Dimensional Box

Consider a spinless quantum particle with mass  $m$  confined to the following one-dimensional box

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{elsewhere} \end{cases} \quad (3)$$

Ordinary Schrödinger equation for such a particle is

$$\frac{P_{op}^2}{2m}\psi(x) = E\psi(x), \quad (4)$$

where  $P_{op}$  is momentum operator. In GUP, existence of minimal length scale leads to the following generalization of momentum operator[22,23]

$$P_{op} = \frac{\hbar}{i} \frac{\partial}{\partial x} [1 + \beta (\frac{\hbar}{i} \frac{\partial}{\partial x})^2], \quad (5)$$

up to first order of  $\beta$ . Now the Schrödinger equation generalizes to

$$\frac{1}{2m}(-\hbar^2 \frac{\partial^2 \psi(x)}{\partial x^2} + 2\beta \hbar^4 \frac{\partial^4 \psi(x)}{\partial x^4}) = E\psi(x), \quad (6)$$

where can be written as

$$2\beta \hbar^4 \frac{\partial^4 \psi(x)}{\partial x^4} - \hbar^2 \frac{\partial^2 \psi(x)}{\partial x^2} - 2mE\psi(x) = 0. \quad (7)$$

Now we want to solve this eigenvalue problem. Existence of minimum observable length means that one cannot have localized states. The very notion of locality and position space representation breaks down in GUP. However as Kempf *et al* have shown[24], when there is no minimal uncertainty in momentum, one can work with the convenient representation of the commutation relations on momentum space wave function. Then one can define states with maximal localization which are proper physical states. One can use them to define a quasi-position representation. This representation has a direct interpretation in terms of position measurements, although it does not diagonalize position operator. Here, since we are dealing with (1) which has no minimum uncertainty in momentum, we can use such quasi-position states,  $\psi(x)$  which are maximally localized.

Now the general solution of equation (7) is

$$\begin{aligned} \psi(x) = & C_1 e^{\frac{1}{2\hbar} \sqrt{\frac{1}{\beta}(1+\sqrt{1+16mE\beta})}x} + C_2 e^{\frac{-1}{2\hbar} \sqrt{\frac{1}{\beta}(1+\sqrt{1+16mE\beta})}x} \\ & + C_3 e^{\frac{i}{2\hbar} \sqrt{\frac{1}{\beta}(\sqrt{1+16mE\beta}-1)}x} + C_4 e^{\frac{-i}{2\hbar} \sqrt{\frac{1}{\beta}(\sqrt{1+16mE\beta}-1)}x}, \end{aligned} \quad (8)$$

where  $C_i$ ,  $i = 1, 2, 3, 4$  are constant coefficients. The general solution is a superposition of  $\sinh x$ ,  $\cosh x$ ,  $\sin x$  and  $\cos x$ . Considering boundary condition in  $x = 0$  leads to the result that  $\cosh x$  and  $\cos x$  should not be considered in physical solution. On the other hand, boundary condition in  $x = a$  implies that  $\sinh x$  should not be present in final solution since in  $x = a$  wave function should be vanishing. Therefore, to satisfy boundary conditions only  $\sin x$  is physically reasonable. It is important to note that in the framework of quantum mechanical perturbational approach, one recovers only  $\sin x$  as leading order solution, which approves the above statement. In other words, considering the term proportional to  $\beta$  in generalized Schrödinger equation as perturbation, one recovers  $\sin x$  as the leading order wave function. Therefore, the solution of generalized Schrödinger equation with the prescribed boundary conditions is

$$\psi(x) = A \sin \left[ \frac{1}{2\hbar} \sqrt{\frac{1}{\beta}(\sqrt{1+16mE\beta}-1)}x \right]. \quad (9)$$

Defining

$$\frac{1}{2\hbar}\sqrt{\frac{1}{\beta}(\sqrt{1+16mE\beta})-1}=j \quad (10)$$

boundary condition in  $x=0$  leads to  $\sin ja=0=\sin n\pi$  where  $n=1,2,3,\dots$ . Therefore  $j=\frac{n\pi}{a}$ . After normalization one finds

$$\psi(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}. \quad (11)$$

Now the energy condition

$$\frac{1}{2\hbar}\sqrt{\frac{1}{\beta}(\sqrt{1+16mE\beta})-1}=\frac{n\pi}{a}, \quad (12)$$

leads to

$$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}+\beta\frac{n^4\pi^4\hbar^4}{ma^4}, \quad (13)$$

up to first order in  $\beta$ . Note that in the framework of GUP up to first order in  $\beta$  there is no change in eigenstates of the quantum particle in box but we have a shift of energy levels equal to  $\beta\frac{n^4\pi^4\hbar^4}{ma^4}$ . When  $\beta\rightarrow 1$ , that is, in extreme quantum gravity limit, this energy shift plays very important role. Figure 1 shows the corresponding situation. As this figure shows increasing  $\beta$  leads to increasing shift in energy levels.

## 4 Momentum Space Wave Function

In ordinary quantum mechanics it is often useful to expand the states  $|\psi\rangle$  in the position eigenbasis  $\{|x\rangle\}$  as  $\langle x|\psi\rangle$ . As has been indicated, in GUP due to the existence of minimal length scale which can not be probed, there are no physical states to perform a position eigenbasis. Although there is a one parameter family of  $x$ -eigenbases, related to the minimal uncertainty in position, but these bases consists of no physical states. Furthermore they could not even be approximated by physical states of increasing localization. However we can still project arbitrary states  $|\phi\rangle$  on maximally localized states  $|\psi_x^{M.L.}\rangle$  to obtain the probability amplitude for the particle being maximally localized around the position  $x$ [24]. In which follows we consider these maximally localized wave functions.

The eigenvalue equation for momentum space wave function in ordinary quantum mechanics is

$$P_{op}u_p(x)=pu_p(x) \quad (14)$$

which  $P_{op}$  stands for momentum operator with  $p$  as its eigenvalue and  $u_p(x)$  as its eigenfunction. Position space representation of  $P_{op}$  is as follows

$$P = \frac{\hbar}{i} \frac{\partial}{\partial x}. \quad (15)$$

Therefore one finds

$$\frac{\hbar}{i} \frac{\partial u_p(x)}{\partial x} = pu_p(x) \quad (16)$$

which has solution such as

$$u_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}. \quad (17)$$

Now we consider gravitational induced uncertainty which leads to the following generalized form for  $P_{op}$

$$P_{op} = \frac{\hbar}{i} \frac{\partial}{\partial x} [1 + \beta (\frac{\hbar}{i} \frac{\partial}{\partial x})^2] \quad (18)$$

up to first order in  $\beta$ . Eigenvalue equation in this case should be modified as follows

$$\frac{\hbar}{i} \frac{\partial}{\partial x} [1 + \beta (\frac{\hbar}{i} \frac{\partial}{\partial x})^2] u_p(x) = pu_p(x), \quad (19)$$

which can be written as

$$\beta \hbar^3 \frac{\partial^3 u_p(x)}{\partial x^3} - \hbar \frac{\partial u_p(x)}{\partial x} + ipu_p(x) = 0. \quad (20)$$

This equation has a complicated solution

$$u_p(x) = C_1 e^{-\frac{i\sqrt{3}(A-12\beta)+A+12\beta}{B}x} + C_2 e^{-\frac{i\sqrt{3}(12\beta-A)+A+12\beta}{B}x} + C_3 e^{\frac{2(A+12\beta)x}{B}}, \quad (21)$$

where  $A$  and  $B$  are complex quantities

$$A = \left( -12[9pi - \sqrt{-\frac{12+81p^2\beta}{\beta}}]\beta^2 \right)^{2/3}$$

and

$$B = 12\beta\hbar A^{1/2}$$

respectively. A lengthy calculation gives the following result for momentum space wave function up to first order of  $\beta$ .

$$u_p(x) = \left( \frac{1-3\beta p^2}{2\pi\hbar} \right)^{1/2} \exp \left( \frac{i(p-\beta p^3)x}{\hbar} \right). \quad (22)$$

It is important to note that this wave function reduces to (17) in the limit of  $\beta \rightarrow 0$  as a consequence of correspondence principle.

## 5 Summary and Conclusions

Although position space representation fails to be valid in the case of GUP due to existence of a minimal observable length, one can consider maximally localized states to obtain the probability amplitude for the particle being maximally localized around the position  $x$ . In this paper, following such a viewpoint, we have calculated eigenfunctions and eigenvalues of a spinless particle confined in an one-dimensional potential box. We have shown that up to first order in GUP parameter, there is no change in eigenfunctions, but there is a shift to energy levels which has considerable effect in the extreme quantum gravity limit  $\beta \rightarrow 1$ . Following the maximally localized picture, we have found momentum space wave function for a free particle. Actually some care should be taken into account regarding the notion of "free particle", since now due to gravitational effect the particle is no longer free.

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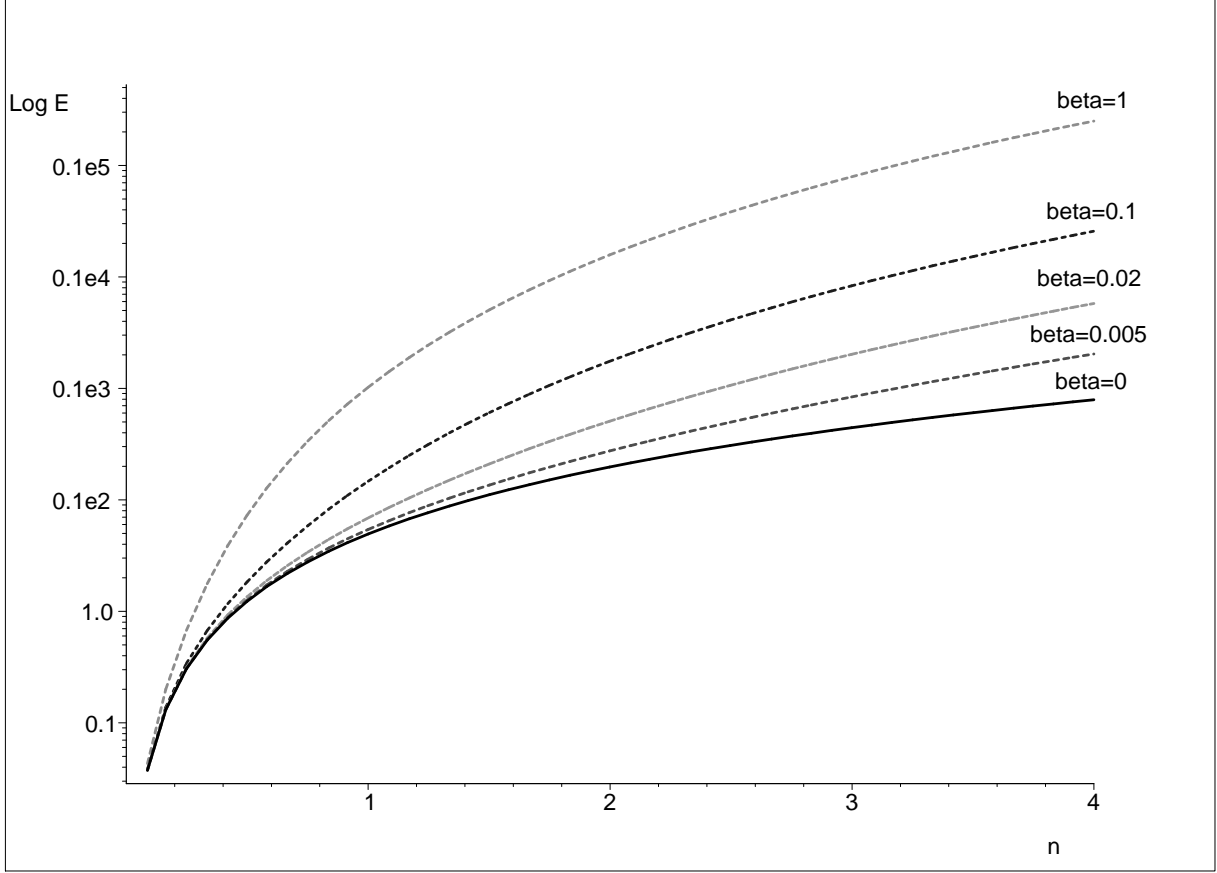


Figure 1: Energy Spectrum of a Particle in One-Dimensional Box in the Framework of GUP. The Effects of Variation in GUP Parameter( $\beta$ ) is Highlighted. To Show the Effect of GUP, we have Considered  $n$  as a Continuous Parameter.  $\beta = 0$  is the Usual Quantum Mechanical Regime.